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CONTINUOUS SPECTRUM FOR SOME CLASSES OF ($p, 2$)-EQUATIONS WITH LINEAR OR SUBLINEAR GROWTH

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Abstract. We are concerned with two classes of nonlinear eigenvalue problems involving equations driven by the sum of the p -Laplace ($p > 2$) and Laplace operators. The main results of this paper establish the existence of a continuous spectrum consisting in an unbounded interval, which is described by using the principal eigenvalue of the Laplace operator.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open bounded set with smooth boundary. A central result in elementary functional analysis and in the linear theory of partial differential equations asserts that the spectrum of the Laplace operator $(-\Delta)$ in $H_0^1(\Omega)$ is *discrete*. More precisely, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \quad (1.1)$$

admits a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \rightarrow +\infty$. The proof of this result relies on the Riesz-Fredholm theory for compact self-adjoint operators (see, e.g., H. Brezis [6, Ch. VI]).

The anisotropic linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x)u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \quad (1.2)$$

was studied starting with the pioneering papers of M. Bocher [5] and P. Hess and T. Kato [11]. We also refer to S. Minakshisundaram and A. Pleijel [14] who proved that problem (1.2) admits an unbounded sequence (λ_n) of eigenvalues, provided that

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V is nonnegative, $V \in L^\infty(\Omega)$ and $V > 0$ in $\omega \subset \Omega$ with $|\omega| > 0$. The case where the weight function V may change sign (that is, V is indefinite) and may have singular points was studied by A. Szulkin and M. Willem [20] who established sufficient conditions for the existence of an unbounded sequence of eigenvalues.

Fix $p \in (1, \infty)$. The quasilinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \quad (1.3)$$

was studied by several mathematicians (see, e.g., A. Anane [1], J. Garcia Azorero and I. Peral Alonso [9], P. Lindqvist [12], A. Szulkin and M. Willem [20]). For instance, A. Anane [1] and P. Lindqvist [12] proved that the first eigenvalue $\lambda = \lambda_1$ of problem (1.3) is *simple* and *isolated* in any bounded domain Ω . By combining topological and variational arguments, A. Szulkin and M. Willem [20] established the existence of a countable family of eigenvalues for a class of quasilinear eigenvalue problems with indefinite weight.

The analysis developed in these papers can be extended to homogeneous eigenvalue problems of the type

$$\begin{cases} -\operatorname{div} A(x, \nabla u) u = \lambda V(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases}$$

where $A(x, \xi) \simeq |\xi|^{p-2} \xi$ fulfills restrictive structural conditions and $V \geq 0$, $V \neq 0$.

In the present paper, we are concerned with the spectral analysis of two classes of $(p, 2)$ -equations, that is, equations driven by the sum of the p -Laplace ($p > 2$) and Laplace operators. These equations describe phenomena arising in mathematical physics. We refer to V. Benci, P. D'Avenia, D. Fortunato and L. Pisani [4] (quantum physics) and L. Cherfils and Y. Ilyasov [7] (plasma physics). Problems involving Laplace operators with different homogeneity have been studied recently by S. Barile and G. Figueiredo [2], D. Motreanu and M. Tanaka [15], N. Papageorgiou and V. Rădulescu [17], N. Papageorgiou, V. Rădulescu and D. Repovš [16], etc.

In comparison with the results described in the first part of this section, the properties established in the present paper deal with a *continuous spectrum* that concentrates at infinity.

2. MAIN RESULTS

Consider the eigenvalue problem

$$\begin{cases} -a \Delta u - b \Delta_p u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \quad (2.1)$$

where a, b are positive real numbers and $p > 2$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a \int_{\Omega} \nabla u \nabla v dx + b \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_{\Omega} u v dx$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.1). Since a and b are positive real numbers, it follows that any eigenvalue λ is positive, too.

Let λ_1 be the *first eigenvalue* (or the *principal frequency*) of the Laplace operator in $H_0^1(\Omega)$, namely the smallest eigenvalue of problem (1.1). The first result of this paper establishes the striking property that the spectrum of problem (2.1) is *continuous*. This description will be performed in terms of λ_1 and does not take into account any contribution of the p -Laplace operator that arises in problem (2.1). More precisely, we prove the following property.

Theorem 1. *Assume that a, b are positive real numbers and $p > 2$.*

Then λ is an eigenvalue of problem (2.1) if and only if $\lambda > a\lambda_1$.

This result shows that the eigenvalues of the nonlinear operator $-a\Delta u - b\Delta_p u$ depend *only* on a and λ_1 . The spectrum is continuous even for $b \rightarrow 0^+$, which corresponds to the case when this operator is “close” to the Laplace operator (hence, with a *discrete* spectrum).

The right-hand side of problem (2.1) is linear. We establish a related continuity property of the spectrum in the case of a suitable linear or sublinear perturbation. In such a case it is not possible to describe the whole spectrum (as done in Theorem 1) but we can assert two facts:

- (i) any $\lambda < a\lambda_1$ cannot be an eigenvalue;
- (ii) all λ sufficiently large is an eigenvalue.

We refer to [13] and [18] for related concentration properties of the spectrum.

Consider the nonlinear problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \quad (2.2)$$

where a, b are positive real numbers and $p > 2$.

We assume that $f : \Omega \rightarrow \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and we set $F(x, t) := \int_0^t f(x, s)ds$.

We suppose that the following hypotheses are fulfilled:

- (f1) we have $|f(x, t)| \leq |t|$ for a.a. $x \in \Omega$, all $t \in \mathbb{R}$;
- (f2) there exists $t_0 \in \mathbb{R}$ such that $F(x, t_0) > 0$ for all $x \in \Omega$;
- (f3) we have $f(x, t) = o(t)$ as $|t| \rightarrow \infty$ uniformly for a.a. $x \in \Omega$.

The following functions satisfy the above assumptions:

- (i) $f(x, t) = V(x) \sin(\alpha t)$, $\alpha > 0$, $V \in L^\infty(\Omega)$, $V > 0$, $\|V\|_{L^\infty} \leq 1$;
- (ii) $f(x, t) = V(x) \log(1 + |t|)$, $V \in L^\infty(\Omega)$, $V > 0$, $\|V\|_{L^\infty} \leq 1$;
- (iii) $f(x, t) = V(x)(|t|^r - |t|^q)$, $0 < q < r < 1$, $V \in L^\infty(\Omega)$, $V > 0$, $\|V\|_{L^\infty} \leq 1$.

We say that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a solution of problem (2.1) if

$$a \int_{\Omega} \nabla u \nabla v dx + b \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_{\Omega} f(x, u) v dx \quad (2.3)$$

whenever $v \in W_0^{1,p}(\Omega)$.

In such a case, the corresponding λ is called an *eigenvalue* of problem (2.2).

Theorem 2. Assume that a, b are positive real numbers, $p > 2$ and hypotheses (f1)-(f3) are fulfilled.

Then any $0 < \lambda \leq a\lambda_1$ is not an eigenvalue of problem (2.2). Moreover, there exists $\lambda^* > 0$ such that all $\lambda > \lambda^*$ is an eigenvalue of problem (2.2).

We do not have any estimate on the value of λ^* . We consider that this is an interesting subject, which should be considered in accordance with the behavior of the nonlinear term f .

The methods developed in this paper allow to consider several classes of differential operators in the left-hand side of problem (2.1), for instance

$$-a\Delta u - b \operatorname{div} \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{(p-2)/2}} \right)$$

or

$$-a\Delta u - b\Delta_p u - b \operatorname{div} \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{(p-2)/2}} \right).$$

We refer for more details to S. Barile and G. Figueiredo [2].

The approach used in this paper can be applied to the abstract framework developed by Mingione *et al.* [3, 8] and corresponding to differential operators of the form

$$-a\Delta_p u - b \operatorname{div} (a(x)|\nabla u|^{q-2}\nabla u) \quad \text{with } 1 < p < q,$$

where $0 \leq a(\cdot) \in C^{0,\alpha}(\overline{\Omega})$.

Notation: for all $u \in W_0^{1,p}(\Omega)$ we denote

$$u_{\pm}(x) := \max\{\pm u(x), 0\}, \quad \text{for } x \in \Omega.$$

By [10, Theorem 7.6] we have $u_{\pm} \in W_0^{1,p}(\Omega)$ and

$$\nabla u_{+} = \begin{cases} \nabla u & \text{on } [u > 0] \\ 0 & \text{on } [u \leq 0] \end{cases} \quad \nabla u_{-} = \begin{cases} \nabla u & \text{on } [u < 0] \\ 0 & \text{on } [u \geq 0]. \end{cases}$$

3. PROOF OF THEOREM 1

We first argue that any $\lambda \leq a\lambda_1$ is not an eigenvalue of problem (2.1). Arguing by contradiction, let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ denote the eigenfunction corresponding to the eigenvalue $\lambda \leq a\lambda_1$. Then

$$a \int_{\Omega} |\nabla u|^2 dx + b \int_{\Omega} |\nabla u|^p dx = \lambda \int_{\Omega} u^2 dx \leq a\lambda_1 \int_{\Omega} u^2 dx. \quad (3.1)$$

Since $p > 2$, it follows that $u \in H_0^1(\Omega) \setminus \{0\}$, hence the variational characterization of λ_1 yields

$$\lambda_1 \int_{\Omega} u^2 dx \leq \int_{\Omega} |\nabla u|^2 dx. \quad (3.2)$$

Combining relations (3.1) and (3.2) we deduce that

$$a\lambda_1 \int_{\Omega} u^2 dx + ab \int_{\Omega} |\nabla u|^p dx \leq a\lambda_1 \int_{\Omega} u^2 dx,$$

a contradiction.

It remains to show that any $\lambda > a\lambda_1$ is an eigenvalue of problem (2.1).

The energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ associated to problem (2.1) is defined by

$$\mathcal{E}(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{2} \int_{\Omega} u^2 dx.$$

We have

$$\mathcal{E}(u) \geq \frac{b}{p} \|u\|_{W_0^{1,p}(\Omega)}^p - \frac{\lambda - a\lambda_1}{2\lambda_1} \|u\|_{H_0^1(\Omega)}^2.$$

Our assumption $p > 2$ implies that

$$\lim_{\|u\|_{W_0^{1,p}(\Omega)} \rightarrow \infty} \mathcal{E}(u) = +\infty,$$

hence \mathcal{E} is coercive.

Consider the minimization problem

$$\inf\{\mathcal{E}(u); u \in W_0^{1,p}(\Omega)\} \quad (3.3)$$

and let (u_n) be a minimizing sequence of (3.3). Since \mathcal{E} is coercive, it follows that (u_n) is bounded. Thus, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega) \subset H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly embedded into $L^2(\Omega)$, we can also assume that

$$u_n \rightarrow u \quad \text{in } L^2(\Omega).$$

Next, using the weakly lower semicontinuity of \mathcal{E} , we deduce that $u \in W_0^{1,p}(\Omega)$ minimizes \mathcal{E} . In order to show that u is nontrivial (hence, an eigenvalue of problem (2.1)), we argue by contradiction and assume that $u = 0$. This implies that \mathcal{E} takes only nonnegative values, so it is enough to prove that

$$\inf\{\mathcal{E}(v); v \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we first choose $w \in C_0^\infty(\Omega)$ such that

$$\lambda_1 < \frac{\int_{\Omega} |\nabla w|^2 dx}{\int_{\Omega} w^2 dx} < \frac{\lambda}{a}. \quad (3.4)$$

This choice is possible due to the hypothesis $\lambda > a\lambda_1$ combined with the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$. We also observe that we have $w \in W_0^{1,p}(\Omega) \setminus \{0\}$. So, for all $t > 0$, we have

$$\mathcal{E}(tw) = \frac{at^2}{2} \int_{\Omega} |\nabla w|^2 dx + \frac{bt^p}{p} \int_{\Omega} |\nabla w|^p dx - \frac{\lambda t^2}{2} \int_{\Omega} w^2 dx$$

$$\begin{aligned}
&= \frac{bt^p}{p} \int_{\Omega} |\nabla w|^p dx + \frac{t^2}{2} \left(a \int_{\Omega} |\nabla w|^2 dx - \lambda \int_{\Omega} w^2 dx \right) \\
&= A \frac{t^2}{2} + B \frac{bt^p}{p},
\end{aligned}$$

where

$$A := a \int_{\Omega} |\nabla w|^2 dx - \lambda \int_{\Omega} w^2 dx < 0$$

and

$$B := \int_{\Omega} |\nabla w|^p dx > 0.$$

Moreover, by the choice of w , cf. (3.4), we have $A < 0$.

In order to obtain $\mathcal{E}(tw) < 0$ it is enough to choose

$$0 < t < \left(-\frac{pA}{2B} \right)^{1/(p-2)}.$$

This completes the proof of Theorem 1. \square

4. PROOF OF THEOREM 2

We first establish that all positive eigenvalues of problem (2.2) are bigger than $a\lambda_1$. Let us observe that relation (2.3) can be rewritten as

$$\begin{aligned}
&a \int_{\Omega} (\nabla u_+ - \nabla u_-) \nabla v dx + b \int_{\Omega} |\nabla u|^{p-2} (\nabla u_+ - \nabla u_-) \nabla v dx = \\
&\lambda \int_{\Omega} (f(x, u_+) + f(x, -u_-)) v dx
\end{aligned} \tag{4.1}$$

whenever $v \in W_0^{1,p}(\Omega)$.

In particular, relation (4.1) shows that $u = e_1$ (namely, the first eigenfunction of the Laplace operator in $H_0^1(\Omega)$) cannot be an eigenvalue of problem (2.2), provided that $\lambda \leq a\lambda_1$.

Taking $v = u_+$ in (4.1) we obtain

$$a \int_{\Omega} |\nabla u_+|^2 dx + b \int_{\Omega} |\nabla u|^{p-2} |\nabla u_+|^2 dx = \lambda \int_{\Omega} f(u_+) u_+ dx. \tag{4.2}$$

Taking $v = u_-$ in (4.1) we obtain

$$a \int_{\Omega} |\nabla u_-|^2 dx + b \int_{\Omega} |\nabla u|^{p-2} |\nabla u_-|^2 dx = -\lambda \int_{\Omega} f(u_-) u_- dx. \tag{4.3}$$

Relations (4.2) and (4.3) in combination with hypothesis (f1) yield, respectively,

$$a\lambda_1 \int_{\Omega} u_+^2 dx \leq a \int_{\Omega} |\nabla u_+|^2 dx \leq \lambda \int_{\Omega} f(u_+) u_+ dx \leq \lambda \int_{\Omega} u_+^2 dx$$

and

$$a\lambda_1 \int_{\Omega} u_-^2 dx \leq a \int_{\Omega} |\nabla u_-|^2 dx \leq -\lambda \int_{\Omega} f(u_+) u_+ dx \leq \lambda \int_{\Omega} u_-^2 dx.$$

Since u is nontrivial, at least one of u_+ or u_- is nontrivial. Thus, the above relations imply that $\lambda \geq a\lambda_1$. Moreover, as we have already observed, $\lambda = a\lambda_1$ cannot be an eigenvalue of problem (2.2), since this would imply that $u = e_1$ is an eigenfunction of problem (2.2), which is impossible. In conclusion, if problem (2.2) admits a solution then $\lambda > a\lambda_1$.

It remains to show that problem (2.2) has a solution for all λ large enough.

The energy functional associated to problem (2.2) is $\mathcal{J} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} F(x, u) dx.$$

Fix $\lambda > a\lambda_1$ (which is a necessary condition for the existence of solutions to problem (2.2)).

Hypothesis (f3) implies that there is a positive constant $C = C(\lambda)$ such that

$$\lambda F(x, u) \leq \frac{a\lambda_1}{2} u^2 + C \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathcal{J}(u) &\geq \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{p} \int_{\Omega} |\nabla u|^p dx - \frac{a\lambda_1}{2} \int_{\Omega} u^2 dx - C|\Omega| \\ &\geq \frac{b}{p} \|u\|_{W_0^{1,p}}^p - C|\Omega|, \end{aligned}$$

hence \mathcal{J} is coercive.

Next, we show that there exists $\lambda^* > 0$ such that

$$\inf\{\mathcal{J}(u); u \in W_0^{1,p}(\Omega)\} < 0.$$

For this purpose we use our assumption (f2) and fix $t_0 \in \mathbb{R}$ such that

$$F(x, t_0) > 0 \quad \text{for all } x \in \Omega.$$

Fix arbitrarily a compact set $K \subset \Omega$ and let $w \in W_0^{1,p}(\Omega)$ such that $w = t_0$ in K and $0 \leq w \leq t_0$ in Ω .

Using hypotheses (f1) it follows that

$$\begin{aligned} \int_{\Omega} F(x, w) dx &= \int_K F(x, w) dx + \int_{\Omega \setminus K} F(x, w) dx \\ &\geq \int_K F(x, t_0) dx - \frac{1}{2} \int_{\Omega \setminus K} w^2 dx \\ &\geq \int_K F(x, t_0) dx - \frac{t_0^2}{2} |\Omega \setminus K|. \end{aligned} \tag{4.4}$$

Relation (4.4) shows that increasing eventually the size of K (in order to have $|\Omega \setminus K|$ small enough) we can assume that

$$\int_{\Omega} F(x, w) dx > 0.$$

We deduce that

$$\mathcal{J}(w) = \frac{a}{2} \int_{\Omega} |\nabla w|^2 dx + \frac{b}{p} \int_{\Omega} |\nabla w|^p dx - \lambda \int_{\Omega} F(x, w) dx < 0,$$

provided that $\lambda > 0$ is large enough. For these values of λ , the energy functional \mathcal{J} has a negative global minimum, hence problem (2.2) admits a solution. This completes the proof. \square

The proof of Theorem 2 shows that we can assume the growth imposed by hypothesis (f3) only on one side, say at $+\infty$:

$$f(x, t) = o(t) \quad \text{as } t \rightarrow +\infty \text{ uniformly for a.a. } x \in \Omega.$$

In such a case, the final part of the proof of Theorem 2 (the existence of λ^*) follows by considering the auxiliary problem

$$\begin{cases} -a\Delta u - b\Delta_p u = \lambda f(x, u_+) & \text{in } \Omega \\ u = 0 & \text{on } \Omega. \end{cases} \quad (4.5)$$

Let u be a solution of problem (4.5). By taking $v = u_-$ as test function we deduce that $u_- = 0$, hence $u \geq 0$. This implies that any solution of (4.5) is also a solution of problem (2.2).

From now on, we follow the same arguments as those developed in the second part of the proof of Theorem 2 by replacing the energy functional \mathcal{J} with $J : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{p} \int_{\Omega} |\nabla u|^p dx - \lambda \int_{\Omega} F(x, u_+) dx.$$

The main result of this paper can be extended in the framework of differential operators with variable exponent; we refer to Rădulescu and Repovš [19] for related results.

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